MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Numerical Integration in 2D

Reading assignment:

Lecture notes, Logan 10.4

Summary:

- Gauss integration on a 2D square domain
- Integration on a triangular domain
- Recommended order of integration
- “Reduced” vs “Full” integration; concept of “spurious” zero energy modes/ “hour-glass” modes

1D quardrature rule recap

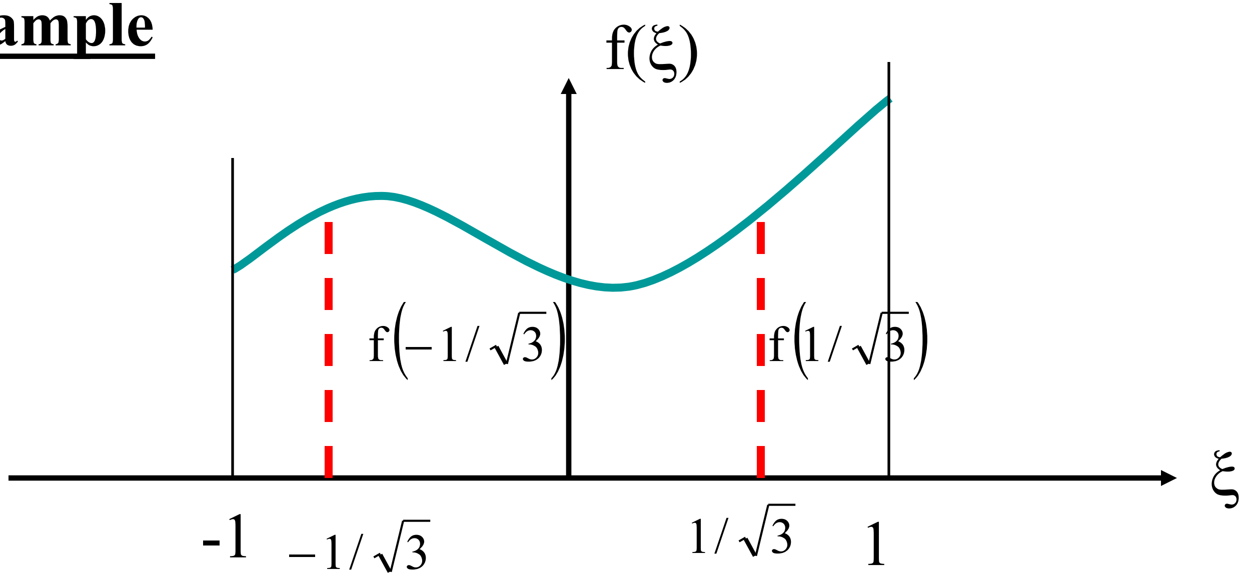
$$\boxed{I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \; f(\xi_i)}$$

Weight Integration point

Choose the integration points **and** weights to maximize accuracy

Newton-Cotes	Gauss quadrature
1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘M-1’ 2. More expensive	1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘2M-1’ 2. Less expensive 3. Exponential convergence, error proportional to $\left(\frac{1}{2M}\right)^{2M}$

Example

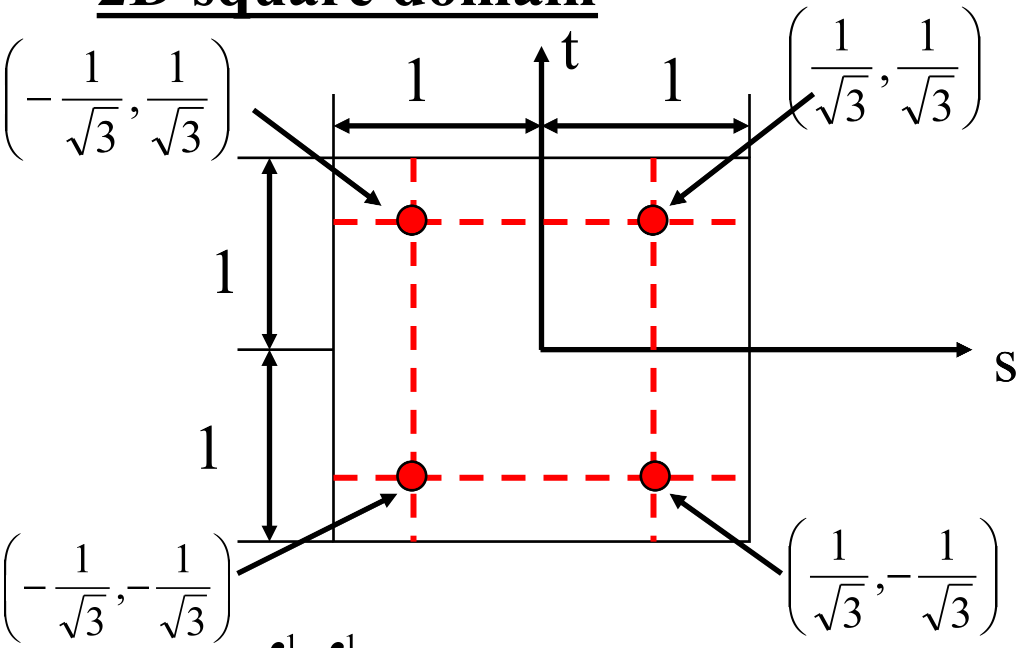


A 2-point Gauss quadrature rule

$$\int_{-1}^{1} f(\xi)\, d\xi \approx f(\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}})$$

is **exact** for a polynomial of degree 3 or less

2D square domain



$$\boxed{I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt$$

$$\approx \int_{-1}^{1}\left(\sum_{j=1}^{M} W_j f(s,t_j)\right) ds$$ Using 1D Gauss rule to integrate along 't'

$$\approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_i W_j f(s_i,t_j)$$ Using 1D Gauss rule to integrate along 's'

$$= \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} f(s_i,t_j)$$ Where $W_{ij} = W_i\, W_j$

For M=2

$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij} f(s_i, t_j)$$

$W_{ij} = W_i\, W_j = 1$

$$= f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$

Number the Gauss points IP=1,2,3,4

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx \sum_{IP=1}^{4} W_{IP} f_{IP}$$

The rule

$$\boxed{I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\; dsdt \approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} f(s_i, t_j)}$$

Uses **M² integration points** on a nonuniform grid inside the parent element and is **exact for a polynomial of degree (2M-1)** i.e.,

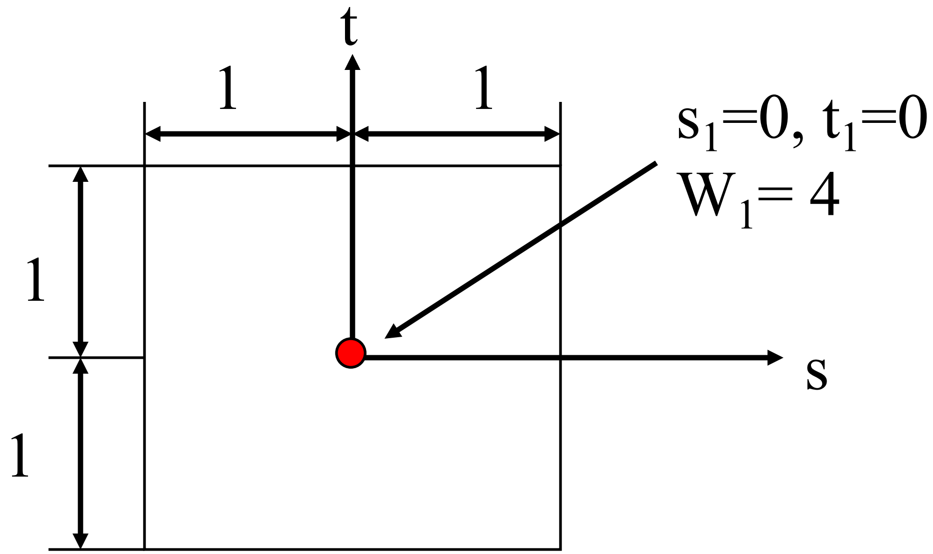
$$\int_{-1}^{1}\int_{-1}^{1} s^{\alpha} t^{\beta} dsdt \overset{exact}{=} \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} s_i^{\alpha} t_j^{\beta} \quad for \;\; \alpha + \beta \leq 2M - 1$$

A **M² –point rule is exact for a complete polynomial of degree (2M-1)**

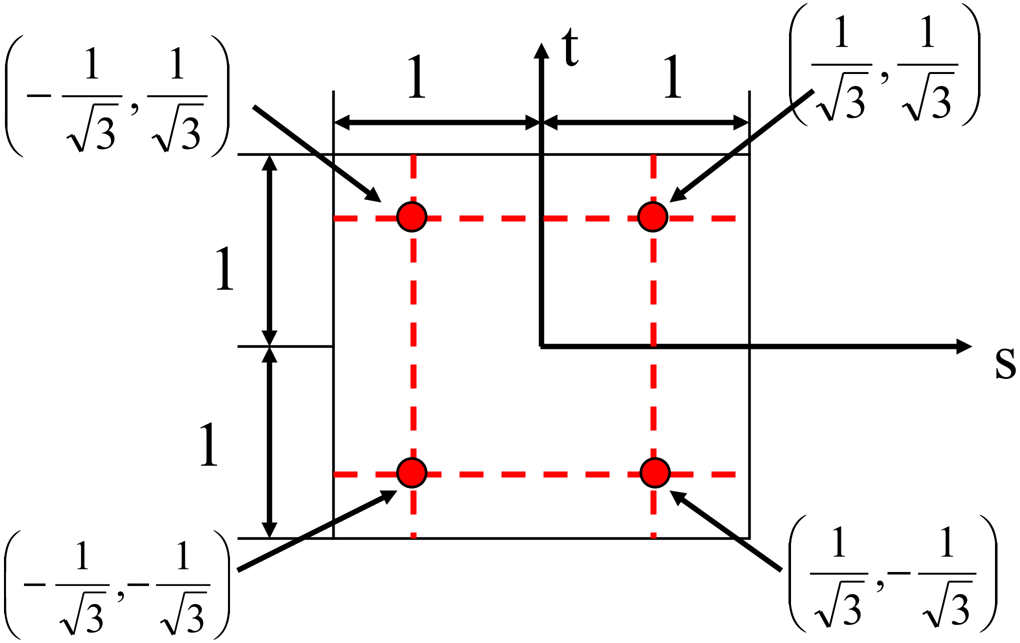
CASE I: M=1 (One-point GQ rule)

$$\boxed{I=\int_{-1}^{1}\int_{-1}^{1} f(s,t)\ dsdt \approx 4f(0,0)}$$

is exact for a product of two linear polynomials



CASE II: M=2 (2x2 GQ rule)

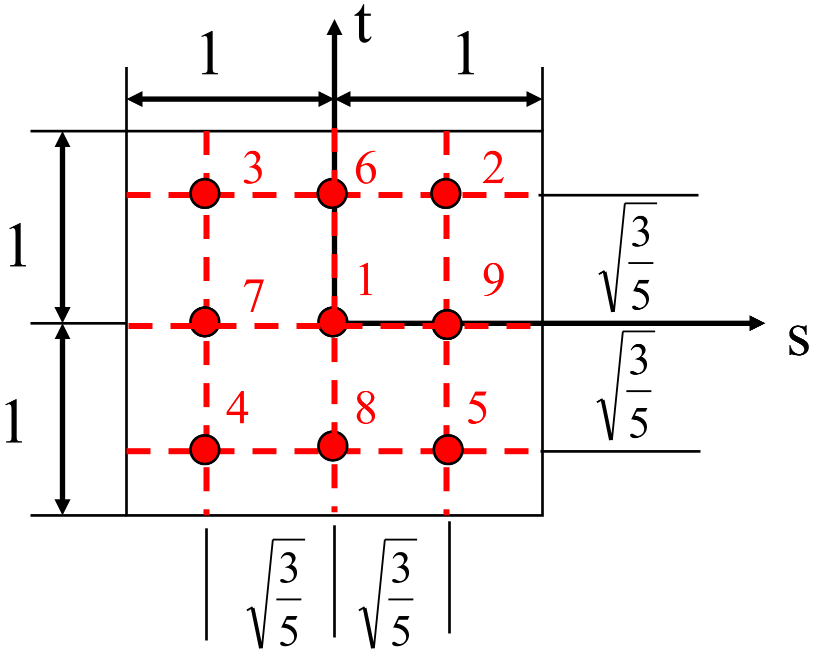


$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij} f(s_i,t_j)$$

$$= f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$$

is exact for a product of two cubic polynomials

CASE III: M=3 (3x3 GQ rule)



$$W_1 = \frac{64}{81},$$

$$W_2 = W_3 = W_4 = W_5 = \frac{25}{81}$$

$$W_6 = W_7 = W_8 = W_9 = \frac{40}{81}$$

$$\boxed{I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx \sum_{i=1}^{3}\sum_{j=1}^{3} W_{ij} f(s_i, t_j)}$$

is exact for a product of two 1D polynomials of degree 5

Examples

If $f(s,t)=1$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 4$$

A **1-point GQ scheme** is sufficient

If $f(s,t)=s$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 0$$

A **1-point GQ scheme** is sufficient

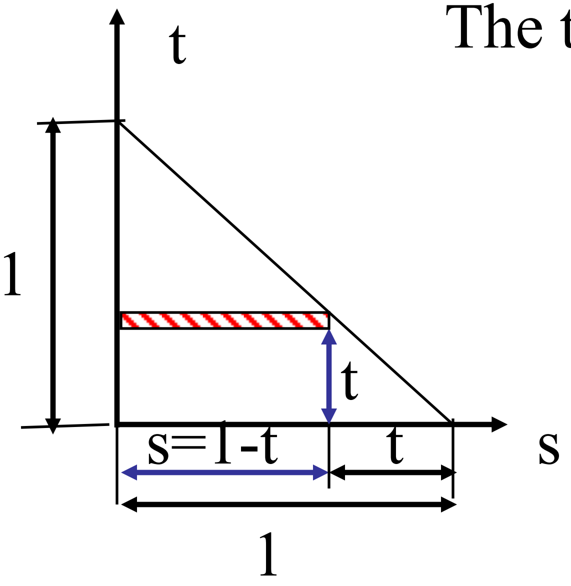
If $f(s,t)=s^2t^2$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = \frac{4}{9}$$

A **3x3 GQ scheme** is sufficient

2D Gauss quadrature for triangular domains

Remember that the **parent element** is a right angled triangle with unit sides



The type of integral encountered

$$\boxed{I = \int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt}}$$

$$I = \int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt}$$

$$\approx \sum_{IP=1}^{M} W_{IP} f_{IP}$$

Constraints on the weights

if f(s,t)=1

$$I=\int_{t=0}^{1}\int_{s=0}^{1-t}\mathrm{f}(s,t)\ \ \mathrm{dsdt}=\frac{1}{2}$$

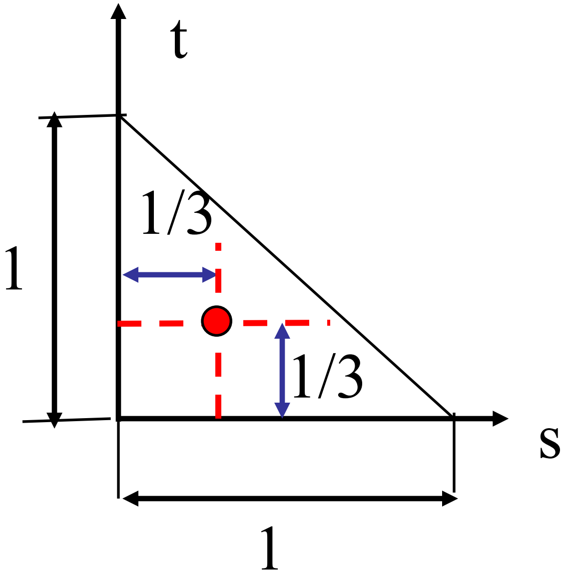
$$=\sum_{IP=1}^{M}W_{IP}$$

$$\therefore\sum_{IP=1}^{M}W_{IP}=\frac{1}{2}$$

Example 1. A M=1 point rule is exact for a polynomial

$$f(s,t) \sim \quad 1$$

$$s \qquad t$$



$$\boxed{I \approx \frac{1}{2} f\left(\frac{1}{3},\frac{1}{3}\right)}$$

Why?

Assume

$$f(s,t)=\alpha_1+\alpha_2 s+\alpha_3 t$$

Then

$$\int_{t=0}^{1}\int_{s=0}^{1-t}\mathrm{f}(s,t)\ \ \mathrm{dsdt}=\frac{1}{2}\alpha_1+\frac{1}{3!}\alpha_2+\frac{1}{3!}\alpha_3$$

But

$$\int_{t=0}^{1}\int_{s=0}^{1-t}\mathrm{f}(s,t)\ \ \mathrm{dsdt}=W_1 f(s_1,t_1)$$

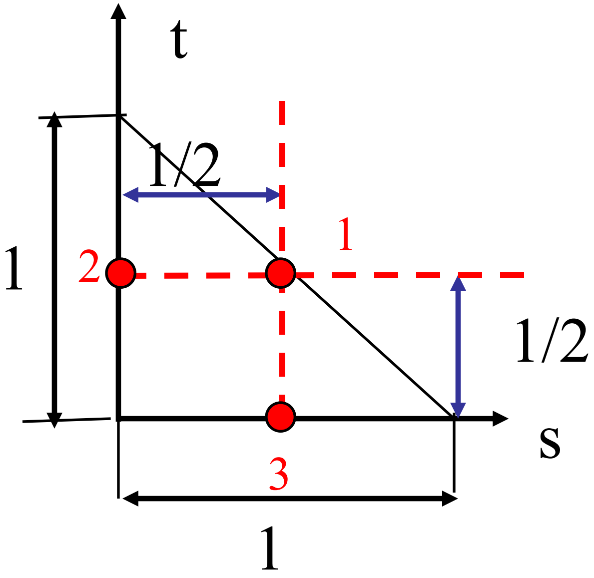
$$\therefore\frac{1}{2}\alpha_1+\frac{1}{3!}\alpha_2+\frac{1}{3!}\alpha_3=W_1\left(\alpha_1+\alpha_2 s_1+\alpha_3 t_1\right)$$

Hence

$$W_1=\frac{1}{2};W_1 s_1=\frac{1}{3!};W_1 t_1=\frac{1}{3!}$$

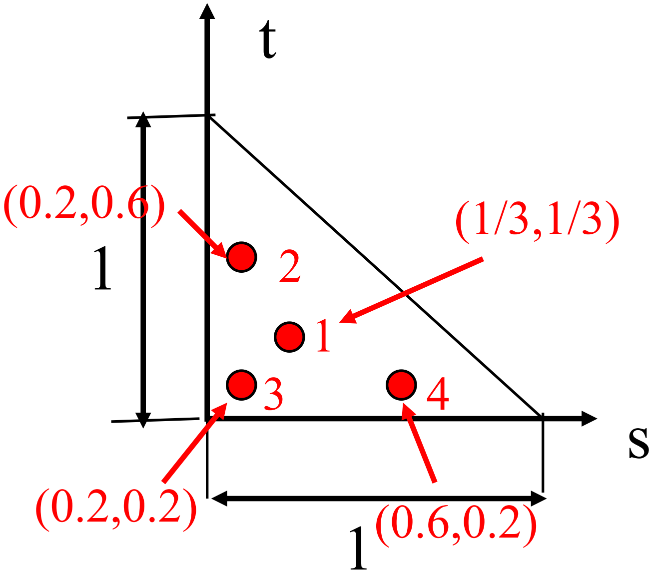
Example 2. A M=3 point rule is exact for a complete polynomial of degree 2

$$f(s,t) \sim \begin{array}{ccc} & 1 & \\ s & & t \\ s^2 & st & t^2 \end{array}$$



$$\boxed{I \approx \frac{1}{6} f\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{6} f\left(\frac{1}{2},0\right) + \frac{1}{6} f\left(0,\frac{1}{2}\right)}$$

Example 4. A M=4 point rule is exact for a complete polynomial of degree 3



$$f(s,t) \sim \quad 1$$

$$s \quad t$$

$$s^2 \quad st \quad t^2$$

$$s^3 \quad s^2t \quad st^2 \quad t^3$$

$$\boxed{I \approx -\frac{27}{96} f\left(\frac{1}{3},\frac{1}{3}\right) + \frac{25}{96} f(0.2,0.6) + \frac{25}{96} f(0.2,0.2) + \frac{25}{96} f(0.6,0.2)}$$

Recommended order of integration

“Finite Element Procedures”

by K. –J. Bathe

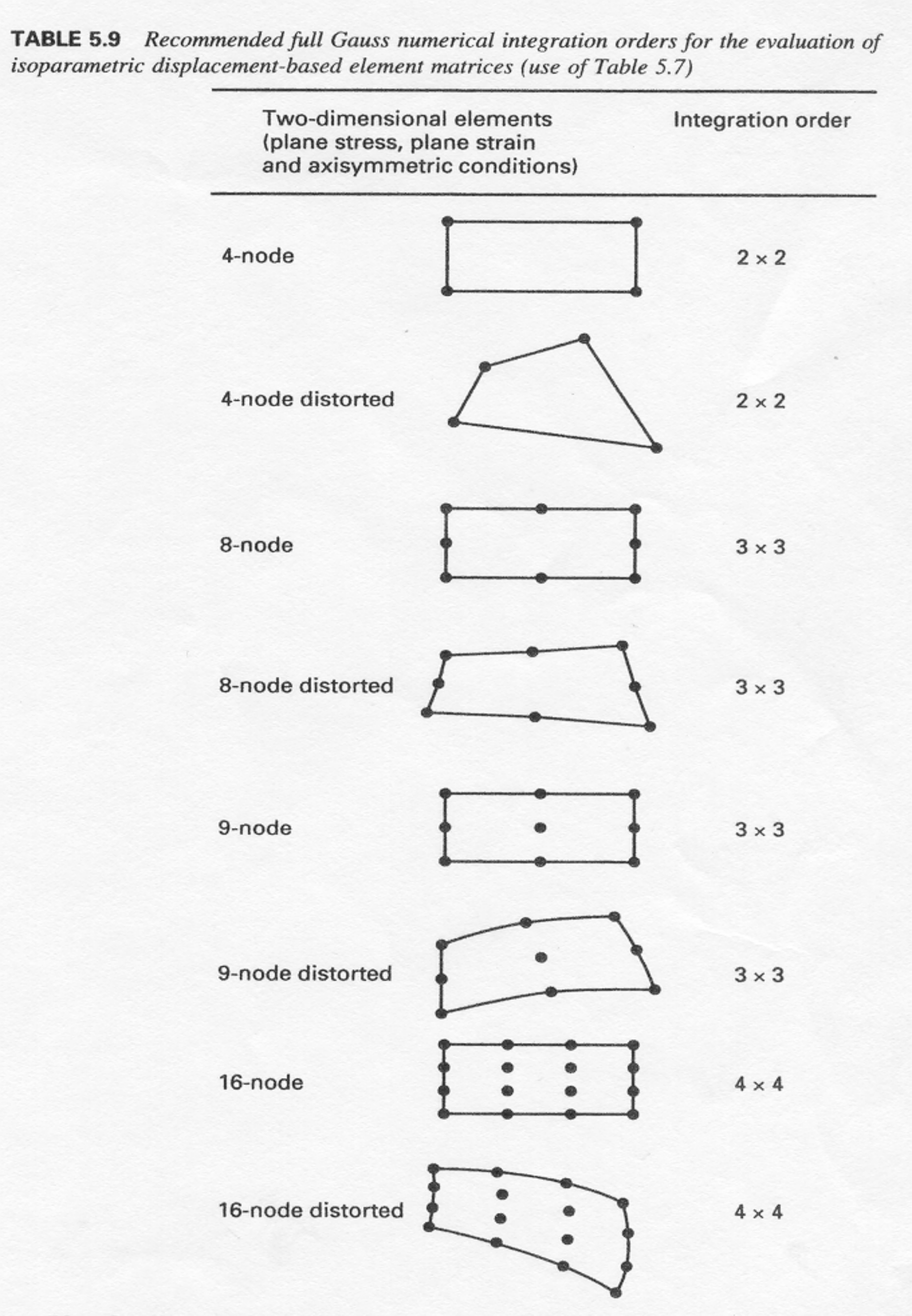


TABLE 5.9 *Recommended full Gauss numerical integration orders for the evaluation of isoparametric displacement-based element matrices (use of Table 5.7)*

Two-dimensional elements (plane stress, plane strain and axisymmetric conditions)	Integration order
4-node	2 × 2
4-node distorted	2 × 2
8-node	3 × 3
8-node distorted	3 × 3
9-node	3 × 3
9-node distorted	3 × 3
16-node	4 × 4
16-node distorted	4 × 4

“Reduced” vs “Full” integration

Full integration: Quadrature scheme sufficient to provide exact integrals of all terms of the stiffness matrix if the element is geometrically undistorted.
Reduced integration: An integration scheme of lower order than required by “full” integration.

Recommendation: Reduced integration is NOT recommended.

Which order of GQ to use for full integration?

To computet the stiffness matrix we need to evaluate the following integral

$$\boxed{\underline{k} = \int_{-1}^{1}\int_{-1}^{1} \underline{\mathrm{B}}^{\mathrm{T}}\, \underline{\mathrm{D}}\, \underline{\mathrm{B}}\, \det(\underline{J})\, \mathrm{dsdt}}$$

For an “undistroted” element det ($\underline{\mathrm{J}}$) =constant

Example : **4-noded parallelogram**

$$N_i \sim \begin{array}{ccc} & 1 & \\ s & & t \\ & st & \end{array}$$

$$\underline{\mathrm{B}} \sim \begin{array}{cc} \multicolumn{2}{c}{1} \\ \mathrm{s} & \mathrm{t} \end{array}$$

$$\underline{\mathrm{B}}^T \underline{\mathrm{D}}\underline{\mathrm{B}} \sim \begin{array}{ccc} & 1 & \\ \mathrm{s} & & \mathrm{t} \\ \mathrm{s}^2 & \mathrm{st} & \mathrm{t}^2 \end{array}$$

Hence, 2M-1=2
M=3/2

Hence we need at least a 2x2 GQ scheme

Example 2: **8-noded Serendipity element**

$N_i \sim$

1
s t
s^2 st t^2
s^2t st^2

$\underline{B} \sim$

1
s t
s^2 st t^2

$$\underline{\underline{B}}^T \underline{\underline{D}}\,\underline{\underline{B}} \sim \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & s & & t & & & \\ & & s^2 & & st & & t^2 & & \\ & s^3 & & s^2t & & st^2 & & t^3 & \\ s^4 & & s^3t & & s^2t^2 & & st^3 & & t^4 \end{array}$$

Hence, 2M-1=4
M=5/2

Hence we need at least a 3x3 GQ scheme

Reduced integration leads to rank deficiency of the stiffness matrix and "spurious" zero energy modes

"Spurious" zero energy mode/ "hour-glass" mode

The strain energy of an element

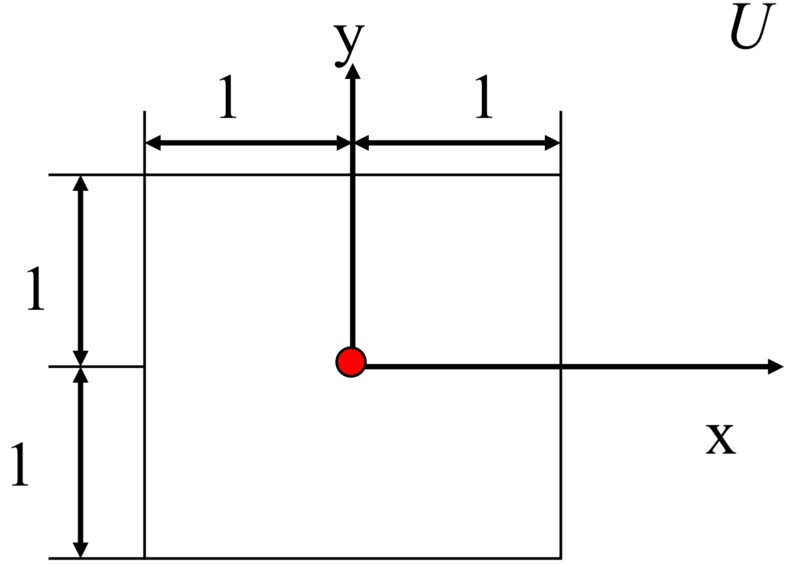
$$U = \frac{1}{2}\underline{d}^T \underline{k}\underline{d} \ = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\, dV$$

Corresponding to a rigid body mode, $\underline{\varepsilon} = \underline{0} \Rightarrow U = 0$

If $U=0$ for a mode $\underline{d}$ that is different from a rigid body mode, then $\underline{d}$ is known as a "spurious" zero energy mode or "hour-glass" mode

Such a mode is **undesirable**

Example 1. 4-noded element



$$U = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\, dV \approx \sum_{i=1}^{NGAUSS} W_i \left(\underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\right)_i$$

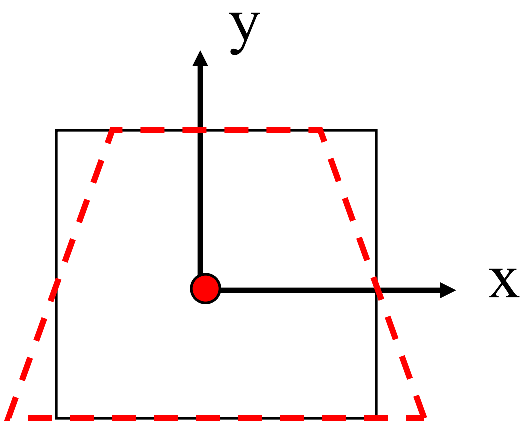
Full integration: NGAUSS=4
Element has 3 zero energy (rigid body) modes

Reduced integration: e.g., NGAUSS=1

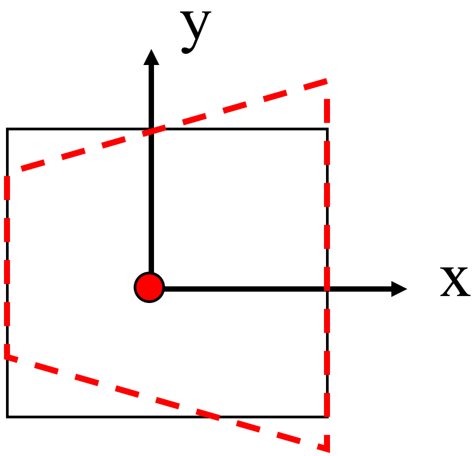
$$U \approx 4\left(\underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\right)_{\substack{x=0\\y=0}}$$

Consider 2 displacement fields

$$u = C\,xy$$
$$v = 0$$



$$u = 0$$
$$v = C\,xy$$

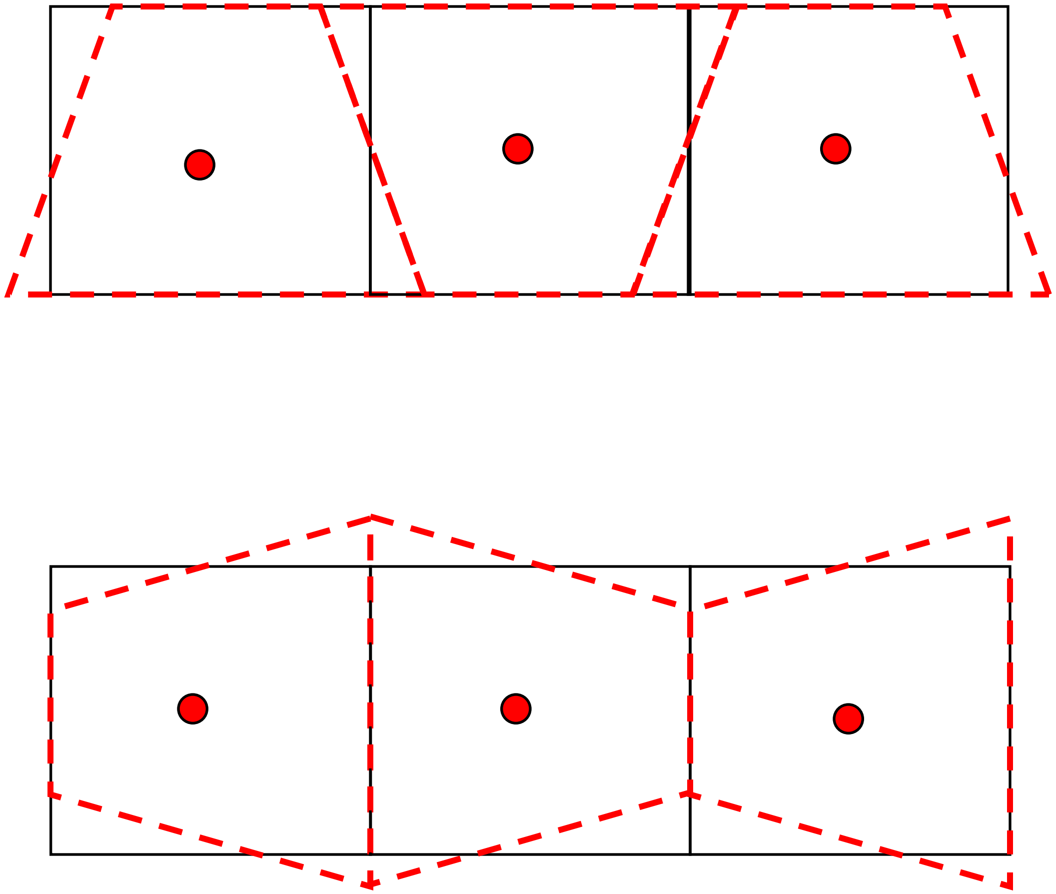


$$At\ x = y = 0 \quad \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$

$$\Rightarrow U = 0$$

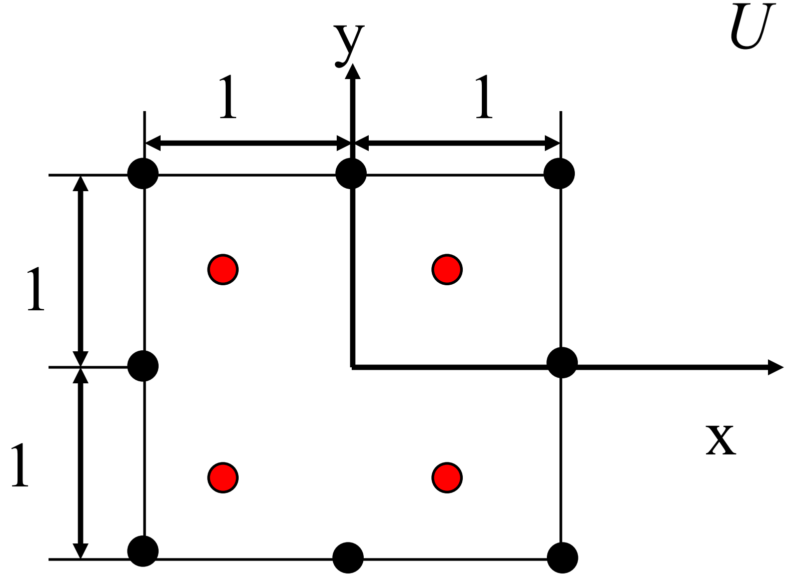
We have therefore 2 hour-glass modes.

Propagation of hour-glass modes through a mesh



Example 2. 8-noded serendipity element

$$U = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\, dV \approx \sum_{i=1}^{NGAUSS} W_i \left(\underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\right)_i$$



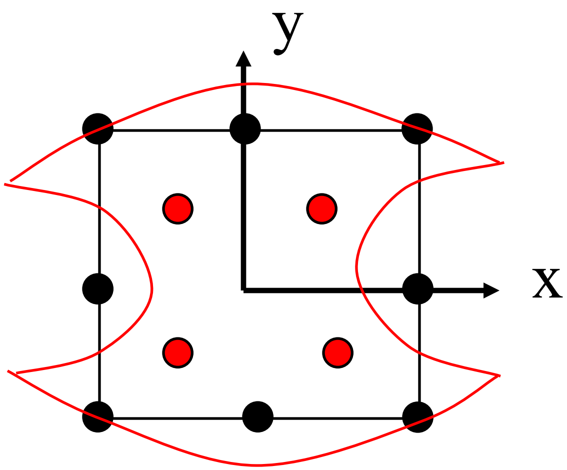
Full integration: NGAUSS=9
Element has 3 zero energy (rigid body) modes

Reduced integration: e.g., NGAUSS=4

Element has one spurious zero energy mode corresponding to the following displacement field

$$u = C\,x\,(y^2 - 1/3)$$

$$v = -C\,y\,(x^2 - 1/3)$$



Show that the strains corresponding to this displacement field are all zero at the 4 Gauss points

Elements with zero energy modes introduce uncontrolled errors and should NOT be used in engineering practice.