ENGN 2220: Mechanics of Solids
Handout on isotropic, linear elastic constitutive equations

In terms of the Lamé moduli, G and λ

Stress in terms of strain:

$$\boldsymbol{\sigma} = 2G\boldsymbol{\epsilon} + \lambda\left(\mathrm{tr}\boldsymbol{\epsilon}\right)\mathbf{1} \qquad \sigma_{ij} = 2G\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

Strain in terms of stress:

$$\boldsymbol{\epsilon} = \frac{1}{2G}\left[\boldsymbol{\sigma} - \frac{\lambda}{2G+3\lambda}\left(\mathrm{tr}\boldsymbol{\sigma}\right)\mathbf{1}\right] \qquad \epsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\lambda}{2G+3\lambda}\sigma_{kk}\delta_{ij}\right]$$

In terms of the shear and bulk moduli, G and K

Stress in terms of strain:

$$\boldsymbol{\sigma} = 2G\boldsymbol{\epsilon}' + K\left(\mathrm{tr}\boldsymbol{\epsilon}\right)\mathbf{1} \qquad \sigma_{ij} = 2G\epsilon'_{ij} + K\epsilon_{kk}\delta_{ij}$$

Strain in terms of stress:

$$\boldsymbol{\epsilon} = \frac{1}{2G}\left[\boldsymbol{\sigma} - \frac{3K-2G}{9K}\left(\mathrm{tr}\boldsymbol{\sigma}\right)\mathbf{1}\right] \qquad \epsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{3K-2G}{9K}\sigma_{kk}\delta_{ij}\right]$$

In terms of the Young's modulus E and Poisson's ratio ν

Stress in terms of strain:

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\left[\boldsymbol{\epsilon} + \frac{\nu}{1-2\nu}\left(\mathrm{tr}\boldsymbol{\epsilon}\right)\mathbf{1}\right] \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right]$$

Strain in terms of stress:

$$\boldsymbol{\epsilon} = \frac{1}{E}\left[\left(1+\nu\right)\boldsymbol{\sigma} - \nu\left(\mathrm{tr}\boldsymbol{\sigma}\right)\mathbf{1}\right] \qquad \epsilon_{ij} = \frac{1}{E}\left[\left(1+\nu\right)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$

Expanded:

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right], \qquad & \epsilon_{12} &= \frac{1+\nu}{E}\sigma_{12} \\
\epsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{33}+\sigma_{11}\right)\right], \qquad & \epsilon_{23} &= \frac{1+\nu}{E}\sigma_{23} \\
\epsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11}+\sigma_{22}\right)\right], \qquad & \epsilon_{13} &= \frac{1+\nu}{E}\sigma_{13}
\end{aligned}$$

Isotropic, linear thermoelasticity in terms of E, ν, and α
Stress in terms of strain and temperature:

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\left[\boldsymbol{\epsilon} + \frac{\nu}{1-2\nu}\left(\mathrm{tr}\boldsymbol{\epsilon}\right)\mathbf{1} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\mathbf{1}\right]$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\delta_{ij}\right]$$

Strain in terms of stress and temperature:

$$\boldsymbol{\epsilon} = \frac{1}{E}\left[(1+\nu)\,\boldsymbol{\sigma} - \nu\left(\mathrm{tr}\boldsymbol{\sigma}\right)\mathbf{1}\right] + \alpha\Delta T\mathbf{1} \qquad \epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\,\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] + \alpha\Delta T\delta_{ij}$$

Isotropic, linear thermoelasticity under plane stress conditions $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
Stress in terms of strain and temperature:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} - \frac{E\alpha\Delta T}{1-\nu}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Strain in terms of stress and temperature:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} + \alpha\Delta T\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Isotropic, linear thermoelasticity under plane strain conditions $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$
Stress in terms of strain and temperature:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Strain in terms of stress and temperature:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} + (1+\nu)\alpha\Delta T\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Relations between elastic moduli

	G	K	E	ν	λ
G, K			$\frac{9KG}{3K+G}$	$\frac{3K-2G}{2(3K+G)}$	$\frac{3K\nu}{1+\nu}$
G, E		$\frac{GE}{3(3G-E)}$		$\frac{E-2G}{2G}$	$\frac{G(E-2G)}{3G-E}$
G, ν		$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1+\nu)$		$\frac{2G\nu}{1-2\nu}$
G, λ		$\lambda+\frac{2}{3}G$	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$	
K, E	$\frac{3EK}{9K-E}$			$\frac{3K-E}{6K}$	$\frac{3K(3K-E)}{9K-E}$
K, ν	$\frac{3K(1-2\nu)}{2(1+\nu)}$		$3K(1-2\nu)$		$K-\frac{2}{3}G$
E, ν	$\frac{E}{2(1+\nu)}$	$\frac{E}{3(1-2\nu)}$			$\frac{E\nu}{(1+\nu)(1-2\nu)}$