ENGN 2220: Mechanics of Solids
Handout on curvilinear coordinates

The general equations of continuum mechanics are best expressed in either direct notation, or in rectangular Cartesian components. However, for the solution of particular problems it is often preferable to employ other coordinate systems. The most commonly encountered coordinate systems are

1. The cylindrical coordinate system which is good for solids that are symmetric around an axis.

2. The spherical coordinate system which is used when there is symmetry about a point.

Here we summarize some important vector and tensor operations in these two curvilinear coordinate systems. For detailed derivations of these relations, see Appendix D of Bower, *Applied Mechanics of Solids*.

1 Cylindrical coordinates

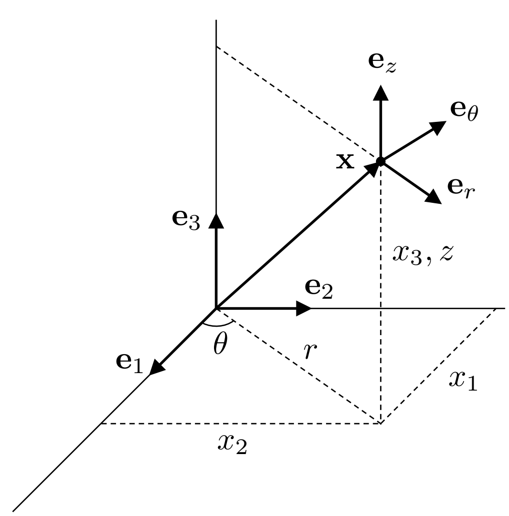
Cylindrical coordinates (r, θ, z) are related to rectangular coordinates (x_1, x_2, x_3) by

$$r = \sqrt{x_1^2 + x_2^2}, \qquad r \geq 0$$
$$\theta = \tan^{-1}(x_2/x_1), \qquad 0 \leq \theta \leq 2\pi$$
$$z = x_3, \qquad -\infty < z < \infty,$$

or inversely by

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z.$$

The orthonormal basis vectors in the cylindrical coordinate system are directed in the radial, $\mathbf{e}_r$, tangential, $\mathbf{e}_\theta$, and axial, $\mathbf{e}_z$, directions as illustrated below.



The position vector is given by

$$\mathbf{r} = r\cos\theta\mathbf{e}_1 + r\sin\theta\mathbf{e}_2 + z\mathbf{e}_3,$$

and the cylindrical basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are related to the Cartesian basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,\\
\mathbf{e}_\theta &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,\\
\mathbf{e}_z &= \mathbf{e}_3.
\end{aligned}$$

Gradient of a scalar field: Consider a scalar field $\psi(r,\theta,z)$,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z.$$

Gradient of a vector field: Consider a vector field $\mathbf{u}(r,\theta,z)$,

$$\begin{aligned}
\nabla\mathbf{u} =& \frac{\partial u_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial u_r}{\partial z}\mathbf{e}_r\otimes\mathbf{e}_z +\\
& \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\otimes\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial u_\theta}{\partial z}\mathbf{e}_\theta\otimes\mathbf{e}_z +\\
& \frac{\partial u_z}{\partial r}\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z,
\end{aligned}$$

or in matrix form

$$[\nabla\mathbf{u}] = \begin{bmatrix}
\dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r} & \dfrac{\partial u_r}{\partial z}\\
\dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} & \dfrac{\partial u_\theta}{\partial z}\\
\dfrac{\partial u_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta} & \dfrac{\partial u_z}{\partial z}
\end{bmatrix}.$$

Divergence of a vector field:

$$\nabla\cdot\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r}$$

Divergence of a tensor field: Consider a tensor field $\mathbf{A}(r,\theta,z)$. The components of the divergence of $\mathbf{A}$ are

$$(\nabla\cdot\mathbf{A})_r = \frac{\partial A_{rr}}{\partial r} + \frac{1}{r}\frac{\partial A_{r\theta}}{\partial\theta} + \frac{\partial A_{rz}}{\partial z} + \frac{1}{r}\left(A_{rr} - A_{\theta\theta}\right)$$

$$(\nabla\cdot\mathbf{A})_\theta = \frac{\partial A_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial\theta} + \frac{\partial A_{\theta z}}{\partial z} + \frac{1}{r}\left(A_{\theta r} + A_{r\theta}\right)$$

$$(\nabla\cdot\mathbf{A})_z = \frac{\partial A_{zr}}{\partial r} + \frac{1}{r}\frac{\partial A_{z\theta}}{\partial\theta} + \frac{\partial A_{zz}}{\partial z} + \frac{A_{zr}}{r}$$

Curl of a vector field:

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r} \right) \mathbf{e}_z$$

Laplacian of a scalar field:

$$\triangle \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}$$

Laplacian of a vector field:

$$\triangle \mathbf{u} = \left(\triangle u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{r^2} u_r \right) \mathbf{e}_r + \left(\triangle u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{1}{r^2} u_\theta \right) \mathbf{e}_\theta + (\triangle u_z)\, \mathbf{e}_z$$

Strain-displacement relations: Let $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z$ denote the components of the displacement vector in the cylindrical coordinate system. The strain is then

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right],$$

and has the following cylindrical components

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \\
\epsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right), \\
\epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right).
\end{aligned}$$

Equilibrium equations: In direct notation, the equilibrium equations are given by

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}.$$

Expanding in cylindrical coordinates,

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = 0, \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + b_\theta = 0 \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + b_z = 0.
\end{aligned}$$

2 Spherical coordinates

Cylindrical coordinates (r, θ, ϕ) are related to rectangular coordinates (x_1, x_2, x_3) by

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad r \geq 0$$

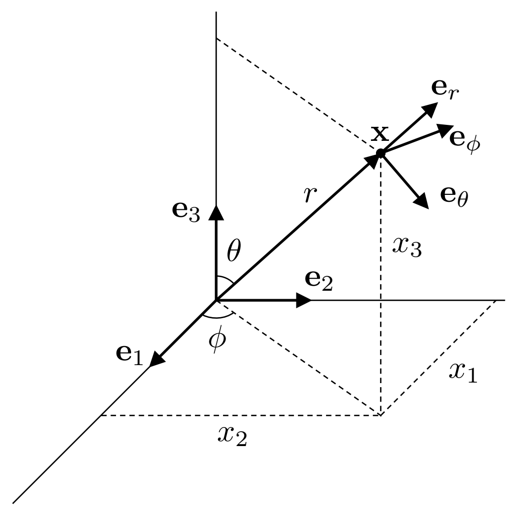
$$\theta = \cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right), \qquad 0 \leq \theta \leq \pi$$

$$\phi = \tan^{-1}(x_2/x_1), \qquad 0 \leq \phi < 2\pi,$$

or inversely by

$$x_1 = r\sin\theta\cos\phi, \qquad x_2 = r\sin\theta\sin\phi, \qquad x_3 = r\cos\theta.$$

The orthonormal basis vectors in the spherical coordinate system $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, directions as illustrated below.



The position vector is given by

$$\mathbf{r} = r\sin\theta\cos\phi\mathbf{e}_1 + r\sin\theta\sin\phi\mathbf{e}_2 + r\cos\theta\mathbf{e}_3,$$

and the spherical basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ are related to the Cartesian basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{e}_1 + \sin\theta\sin\phi\mathbf{e}_2 + \cos\theta\mathbf{e}_3,$$

$$\mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{e}_1 + \cos\theta\sin\phi\mathbf{e}_2 - \sin\theta\mathbf{e}_3,$$

$$\mathbf{e}_\phi = -\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2.$$

Gradient of a scalar field: Consider a scalar field $\psi(r, \theta, \phi)$,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi.$$

Gradient of a vector field: Consider a vector field $\mathbf{u}(r,\theta,\phi)$. In matrix form

$$[\nabla \mathbf{u}] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi} - \dfrac{u_\phi}{r} \\ \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} - \dfrac{\cot\theta}{r}u_\phi \\ \dfrac{\partial u_\phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} + \dfrac{\cot\theta}{r}u_\theta + \dfrac{u_r}{r} \end{bmatrix}.$$

Divergence of a vector field:

$$\nabla\cdot\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{\cot\theta}{r}u_\theta + \frac{2u_r}{r}$$

Divergence of a tensor field: Consider a tensor field $\mathbf{A}(r,\theta,\phi)$. The components of the divergence of $\mathbf{A}$ are

$$(\nabla\cdot\mathbf{A})_r = \frac{\partial A_{rr}}{\partial r} + \frac{1}{r}\frac{\partial A_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_{r\phi}}{\partial \phi} + \frac{1}{r}\left(2A_{rr} - A_{\theta\theta} - A_{\phi\phi} + \cot\theta A_{r\theta}\right)$$

$$(\nabla\cdot\mathbf{A})_\theta = \frac{\partial A_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_{\theta\phi}}{\partial \phi} + \frac{1}{r}\left[2A_{\theta r} + A_{r\theta} + \cot\theta\left(A_{\theta\theta} - A_{\phi\phi}\right)\right]$$

$$(\nabla\cdot\mathbf{A})_\phi = \frac{\partial A_{\phi r}}{\partial r} + \frac{1}{r}\frac{\partial A_{\phi\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{1}{r}\left[2A_{\phi r} + A_{r\phi} + \cot\theta\left(A_{\phi\theta} + A_{\theta\phi}\right)\right]$$

Curl of a vector field:

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \cot\theta\frac{u_\phi}{r}\right)\mathbf{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)\mathbf{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r}\right)\mathbf{e}_\phi$$

Laplacian of a scalar field:

$$\triangle\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{(r\sin\theta)^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}$$

Laplacian of a vector field:

$$\begin{aligned} \triangle\mathbf{u} = &\left[\triangle u_r - \frac{2}{r^2}\left(u_r + u_\theta\cot\theta + \frac{\partial u_\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)\right]\mathbf{e}_r \\ &+ \left[\triangle u_\theta + \frac{1}{r^2}\left(2\frac{\partial u_r}{\partial\theta} - \frac{1}{\sin^2\theta}u_\theta - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)\right]\mathbf{e}_\theta \\ &+ \left[\triangle u_\phi + \frac{1}{r^2\sin\theta}\left(2\frac{\partial u_r}{\partial\phi} + 2\cot\theta\frac{\partial u_\theta}{\partial\phi} - \frac{1}{\sin\theta}u_\phi\right)\right]\mathbf{e}_\phi \end{aligned}$$

Strain-displacement relations: Let $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_\phi \mathbf{e}_\phi$ denote the components of the displacement vector in the spherical coordinate system. The strain then has the following spherical components

$$
\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
\epsilon_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{\cot\theta}{r} u_\theta + \frac{u_r}{r}, \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\
\epsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{\cot\theta}{r} u_\phi\right), \\
\epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right).
\end{aligned}
$$

Equilibrium equations: In spherical coordinates, the equilibrium equations are given by

$$
\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \cot\theta\sigma_{r\theta}\right) + b_r = 0, \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}\left[3\sigma_{r\theta} + \cot\theta(\sigma_{\theta\theta} - \sigma_{\phi\phi})\right] + b_\theta = 0 \\
&\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\cot\theta\sigma_{\theta\phi}) + b_\phi = 0.
\end{aligned}
$$